Weibull-gamma composite distribution: alternative multipath/shadowing fading model

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The Weibull-gamma (WG) distribution, which is appropriate for modelling fading environments when multipath is superimposed on shadowing, is introduced and studied. For this composite distribution the probability density, cumulative distribution, characteristic functions and the moments are derived in closed form. Furthermore, the average bit error and outage probabilities of a receiver operating over WG fading channels are assessed and compared with the corresponding performances obtained using other composite distributions.

Introduction: In many real life propagation links, multipath fading and shadowing occur simultaneously, yielding a composite fading environment [1]. Depending upon the nature of the radio propagation environment, several distributions have been proposed for modelling multipath fading, including Rayleigh, Nakagami and Weibull [1, 2], while shadowing is modelled by the lognormal distribution [1]. Based on these distributions various combinations have been suggested for obtaining the composite environment, including the widely accepted Rayleigh- and Nakagami-lognormal [1] and a relatively new one, the Weibull-lognormal (WL) [3]. A common characteristic of these lognormal-based composite distributions is their complicated mathematical form that limits their potential applications. An alternative approach is to employ the mathematically more convenient gamma distribution for accurately modelling the shadowing effects [4]. Hence, using the gamma distribution, simpler composite distributions have been proposed, such as the K [4] and the generalised-K [5]. Furthermore, in a recent work [6], the probability density function (pdf) of the Weibull-gamma (WG) fading model was presented in terms of the generalised hypergeometric function [6, eqn (5)]. However, to the best of the author's knowledge, a thorough analysis of the WG statistical characteristics is not available in the open technical literature and thus is the subject of the current work. Therefore, in this Letter, by deriving an alternative to the [6, eqn (5)] representation for the pdf, the cumulative distribution function (cdf), the characteristic function (CF) and the moments of the WG composite distribution are presented. Capitalising on these results, the performance of a single receiver operating over such fading/shadowing channels is analysed.

Weibull-gamma statistical characteristics: Let R represent the channel fading envelope following the Weibull distribution with pdf given by [1]

$$f_R(x) = \beta\left[\frac{\Gamma(1+2/\beta)}{y}\right]^{\beta/2} x^{\beta-1} \exp\left[-\left(x^2\Gamma\left(1+\frac{2}{\beta}\right)\right)^{\beta/2} y^{-\beta/2}\right], x \geq 0 \tag{1}$$

where β is distribution's shaping parameter, y is the average fading power $y = E\langle R^2\rangle$, with $E\langle\cdot\rangle$ denoting expectation, and $\Gamma(\cdot)$ is the gamma function [7, eqn (8.310/1)]. When multipath fading is superimposed on shadowing, y slowly varies and hence it can be considered as a random variable following the gamma distribution with pdf given by

$$f_y(y) = \frac{y^{\alpha-1}\exp(-y/\Omega)}{\Gamma(\alpha)\Omega^{\alpha}}, y \geq 0 \tag{2}$$

where $\alpha > 0$ is the shaping parameter and $\Omega = \mathrm{E}\langle y^2\rangle$. Under these circumstances the pdf in (1) is conditioned on y, and in order to remove this conditioning (1) is averaged over (2) as

$$f_X(x) = \int_0^{\infty} f_{R|y}(x|y) f_y(y) dy \tag{3}$$

Using the Meijer-G representation for the exponentials [8, eqn (11)], [7, eqn (9.31/2)], [8, eqn (21)], and after some mathematical manipulations, the pdf of the WG composite distribution can be expressed in closed form as

$$f_X(x) = \frac{\beta x^{\beta-1}}{\Gamma(\alpha)}\left[\frac{\Gamma(1+2/\beta)}{\Omega}\right]^{\beta/2} \frac{\kappa^{1/2}\lambda^{\alpha-\frac{\beta+1}{2}}}{(2\pi)^{\frac{\lambda+\kappa}{2}-1}} \times G_{0,\kappa+\lambda}^{\kappa+\lambda,0}\left(\frac{\left[x^2\Gamma(1+2/\beta)\right]^{\lambda}}{\Omega^{\lambda}\kappa^{\kappa}\lambda^{\lambda}}\middle|\begin{matrix}-\\ b_{\kappa+\lambda}\end{matrix}\right) \tag{4}$$

where $G(\cdot)$ is the Meijer's G-function [7, eqn (9.301)], $b_{\kappa+\lambda} = 1-\Delta(\kappa, 1), 1-\Delta(\lambda, 1-\alpha+\beta/2)$ with $\Delta(x, y) = y/x, (y+1)/x, \ldots, (y+x-1)/x$, while λ and κ are positive integers properly chosen in order to satisfy $\lambda/\kappa = \beta/2$. By varying α and β, the pdf of this generalised-Weibull distribution can simultaneously describe several multipath and shadowing conditions. For instance, for $\alpha \to \infty$, (4) approximates the Weibull distribution, for $\beta = 2$ and using [7, eqn (9.34/3)], it simplifies to the K-distribution and hence approximates the Rayleigh-lognormal distribution, while for $\beta \to \infty, \alpha \to \infty$, (4) approaches the additive white Gaussian noise (AWGN) channel. Furthermore, by following a similar procedure as in [4], a relationship between the parameters α and β of the WG distribution and the mean, μ, and standard deviation, σ, of the WL can be derived as

$$\mu = \frac{10}{\ln(10)}[\ln(\Omega) + \Psi(\alpha)], \sigma = \frac{10}{\ln(10)}\sqrt{\Psi'(\alpha)} \tag{5}$$

where $\Psi(\cdot)$ and $\Psi'(\cdot)$ are the psi function and its derivative, respectively [7, eqn (8.360/1)]. Substituting (4) in the definition of the nth order moments of X, $\mu_X(n) \triangleq E\langle X^n\rangle$, making a change of variables and using [7, eqn (7.811/4)], a closed-form expression for $\mu_X(n)$ can be obtained as

$$\mu_X(n) = \frac{\beta}{2}\left[\frac{\Gamma(1+2/\beta)}{\Omega}\right]^{-n/2} \frac{\kappa^{n/\beta+3/2}\lambda^{a+n/2-3/2}}{(2\pi)^{\frac{\lambda+\kappa}{2}-1}\Gamma(\alpha)} \times \prod_{i=1}^{\kappa+\lambda}\Gamma\left(b_i + \frac{\beta+n}{2\lambda}\right) \tag{6}$$

Furthermore, using the product theorem for the gamma functions [7, eqn (8.335)] and after some mathematical manipulations, (6) simplifies to

$$\mu_X(n) = \frac{\Gamma(1+n/\beta)\Gamma(\alpha+n/2)}{\Gamma(\alpha)}\left[\frac{\Gamma(1+2/\beta)}{\Omega}\right]^{-n/2} \tag{7}$$

Substituting (4) in the definition of the cdf of X, $F_X(x) \triangleq \int_0^x f_X(x)dx$, making a change of variables and using [8, eqn (26)] yields

$$F_X(x) = \frac{\beta}{2}\left[\frac{\Gamma(1+2/\beta)}{\Omega}\right]^{\beta/2} \frac{\kappa^{1/2}\lambda^{\alpha-(\beta+3)/2}x^{\beta}}{\Gamma(\alpha)(2\pi)^{(\lambda+\kappa)/2-1}} \times G_{1,\kappa+\lambda+1}^{\kappa+\lambda,1}\left(\frac{\left[x^2\Gamma(1+2/\beta)\right]^{\lambda}}{\Omega^{\lambda}\kappa^{\kappa}\lambda^{\lambda}}\middle|\begin{matrix}1-1/\kappa\\ b_{\kappa+\lambda}, -1/\kappa\end{matrix}\right) \tag{8}$$

The CF of X is defined as $\Phi_X(s) \triangleq E\langle\exp(-jsX)\rangle$, where $j = \sqrt{-1}$. Starting from this definition, using (4) and [8, eqn (11) and (21)], $\Phi_X(s)$ can be obtained as

$$\Phi_X(s) = \left[\frac{\Gamma(1+2/\beta)}{\Omega(js)^2}\right]^{\beta/2} \frac{\beta 2^{\beta-1/2}\kappa^{1/2}\lambda^{\alpha+\beta/2-1}}{(2\pi)^{(3\lambda+\kappa-3)/2}\Gamma(\alpha)} \times G_{2\lambda,\kappa+\lambda}^{\kappa+\lambda,2\lambda}\left(\frac{\left[2^2\Gamma(1+2/\beta)\lambda\right]^{\lambda}}{\left[\Omega(js)^2\right]^{\lambda}\kappa^{\kappa}}\middle|\begin{matrix}\Delta(2\lambda, 1-\beta)\\ b_{\kappa+\lambda}\end{matrix}\right) \tag{9}$$

Single receiver: Let us consider a receiver operating over WG composite fading channels. The equivalent baseband received signal can be expressed as $r = sh + n$, where s is the transmitted complex symbol with energy $E_s = \mathrm{E}\langle|s|^2\rangle$, n is the complex AWGN with single-sided power spectral density N_0 and h is the channel complex gain, i.e. $X = |h|$. The instantaneous signal-to-noise ratio (SNR) per received symbol is given by $\gamma = X^2E_s/N_0$, while using (7), i.e. setting $n = 2$, the corresponding average SNR can be obtained as $\overline{\gamma} = E\langle X^2\rangle E_s/N_0 = \Omega\alpha E_s/N_0$. Using the last two expressions in (8), the cdf of γ can be easily obtained as

$$F_\gamma(\gamma) = \left[\frac{\Gamma(1+2/\beta)}{\bar{\gamma}/\alpha}\right]^{\beta/2} \frac{\beta\kappa^{1/2}\lambda^{\alpha-(\beta+3)/2}\gamma^{\beta/2}}{2\Gamma(\alpha)\left(\sqrt{2\pi}\right)^{\lambda+\kappa-2}} \times G^{\kappa+\lambda,1}_{1,\kappa+\lambda+1}\left(\frac{[\alpha\Gamma(1+2/\beta)\gamma]^\lambda}{(\bar{\gamma}\lambda)^\lambda\kappa^\kappa}\middle| \begin{matrix} 1-1/\kappa \\ b_{\kappa+\lambda}, -1/\kappa \end{matrix}\right) \tag{10}$$

For the receiver under consideration, the outage probability, P_{out}, defined as the probability that γ falls below a certain specified threshold, γ_{th}, can be easily obtained, using (10), as $P_{out} = F_\gamma(\gamma_{th})$. Moreover, by making a change of variables in (4) and following a similar procedure as for deriving (9), the moments generating function (MGF) of γ, defined as $M_\gamma(s) \triangleq E\langle\exp(-s\gamma)\rangle$, can be obtained as

$$M_\gamma(s) = \left[\frac{\Gamma(1+2/\beta)}{\bar{\gamma}s/\alpha}\right]^{\beta/2} \frac{\beta\kappa^{1/2}\lambda^{\alpha-1}/2}{\Gamma(\alpha)\left(\sqrt{2\pi}\right)^{2\lambda+\kappa-3}} \times G^{\kappa+\lambda,\lambda}_{\lambda,\kappa+\lambda}\left(\frac{[\alpha\Gamma(1+2/\beta)]^\lambda}{(\bar{\gamma}s)^\lambda\kappa^\kappa}\middle| \begin{matrix} \Delta(\lambda, 1-\beta/2) \\ b_{\kappa+\lambda} \end{matrix}\right) \tag{11}$$

By using (11) and following the MGF-based approach, direct calculation of the average bit error probability (ABEP) for non-coherent binary frequency shift keying and differential binary phase shift keying (DBPSK) is possible [1]. For example, the ABEP of DBPSK is given by $P_{be} = 0.5M_\gamma(1)$ [1].

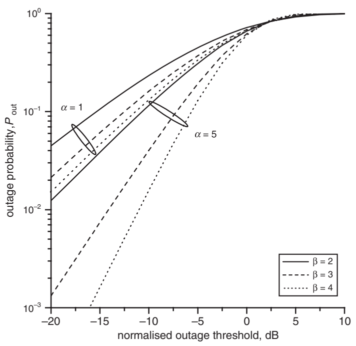

Fig. 1 *Outage probability against normalised outage threshold for several values of α and β*

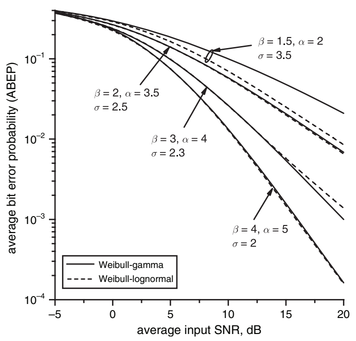

Fig. 2 *ABEP of WG and WL fading channels against average input SNR for several values of a, β and σ*

Numerical results: In Fig. 1, the P_{out} is plotted against the normalised outage threshold, $\gamma_{th}/\bar{\gamma}$, for several values of α and β. It is shown that as $\gamma_{th}/\bar{\gamma}$ and/or α, β increase the outage performance improves. It should be noted that the curves for $\beta = 2$ correspond to K fading channels, while it can be easily observed that the gap among the curves increases as α increases. Fig. 2 shows the ABEP of a single receiver operating over the 'equivalent' WG and WL composite fading channels against $\bar{\gamma}$. The ABEP for WG fading channels is determined using (11), while for the WL numerical integration techniques have been employed. Moreover, in order to obtain equivalent channel conditions, (5) is used for selecting appropriate values for α and σ. In Fig. 2, it is evident that the ABEP performances for these fading channel models are very close, and this close agreement further improves as α increases. It is noted that similar ABEP performance results have been obtained for other modulation schemes, e.g. M-ary PSK.

Conclusions: In this Letter the most important statistical characteristics, such as pdf, cdf, CF and moments of the WG composite distribution have been derived. Using these results, the performance analysis of a receiver operating over WG fading channels has been investigated in terms of the ABEP and the P_{out}. Numerically evaluated results have confirmed that the WG distribution can be used as an alternative to the common, but much more complex, employed Rayleigh- and Weibull-lognormal distributions as a reliable ABEP predictor in multipath/shadowing fading channels.

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